Logic

Logic, the study of correct reasoning, especially as it involves the drawing of inferences.

This article discusses the basic elements and problems of contemporary logic and provides an overview of its different fields. For treatment of the historical development of logic, *see* logic, history of. For detailed discussion of specific fields, *see* the articles applied logic, formal logic, modal logic, and logic, philosophy of.

Scope And Basic Concepts

An inference is a rule-governed step from one or more propositions, called premises, to a new proposition, usually called the conclusion. A rule of inference is said to be truth-preserving if the conclusion derived from the application of the rule is true whenever the premises are true. Inferences based on truth-preserving rules are called deductive, and the study of such inferences is known as deductive logic (*see* formal logic). An inference rule is said to be valid, or deductively valid, if it is necessarily truth-preserving. That is, in any conceivable case in which the premises are true, the conclusion yielded by the inference rule will also be true. Inferences based on valid inference rules are also said to be valid.

Logic in a narrow sense is equivalent to deductive logic. By definition, such reasoning cannot produce any information (in the form of a conclusion) that is not already contained in the premises. In a wider sense, which is close to ordinary usage, logic also includes the study of inferences that may produce conclusions that contain genuinely new information. Such inferences are called ampliative or inductive, and their formal study is known as inductive logic. They are illustrated by the inferences drawn by clever detectives, such as the fictional Sherlock Holmes.

The contrast between deductive and ampliative inferences may be illustrated in the following examples. From the premise "somebody envies everybody," one can validly infer that "everybody is envied by somebody." There is no conceivable case in which the

premise of this inference is true and the conclusion false. However, when a forensic scientist infers from certain properties of a set of human bones the approximate age, height, and sundry other characteristics of the deceased person, the reasoning used is ampliative, because it is at least conceivable that the conclusions yielded by it are mistaken.

In a still narrower sense, logic is restricted to the study of inferences that depend only on certain logical concepts, those expressed by what are called the "logical constants" (logic in this sense is sometimes called elementary logic). The most important logical constants are quantifiers, propositional connectives, and identity. Quantifiers are the formal counterparts of English phrases such as "there is …" or "there exists …," as well as "for every …" and "for all …" They are used in formal expressions such as ($\exists x$) (read as "there is an individual, call it x, such that it is true of x that …") and ($\forall y$) (read as "for every individual, call it y, it is true of y that …"). The basic propositional connectives are approximated in English by "not" (~), "and" (&), "or" ($\vee$), and "if … then …" ($\supset$). Identity, represented by $\equiv$, is usually rendered in English as "… is …" or "… is identical to …" The two example propositions above can then be expressed as (1) and (2), respectively:

> *(1) ($\exists x$)($\forall y$) (x envies y)*
>
> *(2) ($\forall y$)($\exists x$) (x envies y)*

The way in which the different logical constants in a proposition are related to each other is known as the proposition's logical form. Logical form can also be thought of as the result of replacing all of the nonlogical concepts in a proposition by logical constants or by general logical symbols known as variables. For example, by replacing the relational expression "a envies b" by "E(a,b)" in (1) and (2) above, one obtains (3) and (4), respectively:

> *(3) ($\exists x$)($\forall y$) E(x,y)*
>
> *(4) ($\forall y$)($\exists x$) E(x,y)*

The formulas in (3) and (4) above are explicit representations of the logical forms of the corresponding English propositions. The study of the relations between such uninterpreted formulas is called formal logic.

It should be noted that logical constants have the same meaning in logical formulas, such as (3) and (4), as they do in propositions that also contain nonlogical concepts, such as (1) and (2). A logical formula whose variables have been replaced by nonlogical concepts

(meanings or referents) is called an "interpreted" proposition, or simply an "interpretation." One way of expressing the validity of the inference from (3) to (4) is to say that the corresponding inference from a proposition like (1) to a proposition like (2) will be valid for all possible interpretations of (3) and (4).

Valid logical inferences are made possible by the fact that the logical constants, in combination with nonlogical concepts, enable a proposition to represent reality. Indeed, this representational function may be considered their most fundamental feature. A proposition G, for example, can be validly inferred from another proposition F when all of the scenarios represented by F—the scenarios in which F is true—are also scenarios represented by G—the scenarios in which G is true. In this sense, (2) can be validly inferred from (1) because all of the scenarios in which it is true that someone envies everybody are also scenarios in which it is true that everybody is envied by at least one person.

A proposition is said to be logically true if it is true in all possible scenarios, or "possible worlds." A proposition is contradictory if it is false in all possible worlds. Thus, another way to express the validity of the inference from F to G is to say that the conditional proposition "If F, then G" (F ⊃ G) is logically true.

Not all philosophers accept these explanations of logical validity, however. For some of them, logical truths are simply the most general truths about the actual world. For others, they are truths about a certain imperceptible part of the actual world, one that contains abstract entities like logical forms.

In addition to deductive logic, there are other branches of logic that study inferences based on notions such as knowing that (epistemic logic), believing that (doxastic logic), time (tense logic), and moral obligation (deontic logic), among others. These fields are sometimes known collectively as philosophical logic or applied logic. Some mathematicians and philosophers consider set theory, which studies membership relations between sets, to be another branch of logic.

Logical Notation

The way in which logical concepts and their interpretations are expressed in natural languages is often very complicated. In order to reach an overview of logical truths and valid inferences, logicians have developed various streamlined notations. Such notations can be thought of as artificial languages when their nonlogical concepts are interpreted; in this respect they are comparable to computer languages, to some of which they are in

fact closely related. The propositions (1)-(4) illustrate one such notation.

Logical languages differ from natural ones in several ways. The task of translating between the two, known as logic translation, is thus not a trivial one. The reasons for this difficulty are similar to the reasons why it is difficult to program a computer to interpret or express sentences in a natural language.

Consider, for example, the sentence

> *(5) If Peter owns a donkey, he beats it.*

Arguably, the logical form of (5) is

> *(6) (∀x)[(D(x) & O(p,x) ⊃ B(p,x)]*

where D(*x*) means "*x* is a donkey," O(*x*,*y*) means "*x* owns *y*," B(*x*,*y*) means "*x* beats *y*," and "p" refers to Peter. Thus (6) can be read: "For all individuals *x*, if *x* is a donkey and Peter owns *x*, then Peter beats *x*. Yet theoretical linguists have found it extraordinarily difficult to formulate general translation rules that would yield a logical formula such as (6) from an English sentence such as (5).

Contemporary forms of logical notation are significantly different from those used before the 19th century. Until then, most logical inferences were expressed by means of natural language supplemented with a smattering of variables and, in some cases, by traditional mathematical concepts. One can in fact formulate rules for logical inferences in natural languages, but this task is made much easier by the use of a formal notation. Hence, from the 19th century on most serious research in logic has been conducted in what is known as symbolic, or formal, logic. The most commonly used type of formal logical language was invented by the German mathematician Gottlob Frege (1848–1925) and further developed by the British philosopher Bertrand Russell (1872–1970) and his collaborator Alfred North Whitehead (1861–1947) and the German mathematician David Hilbert (1862–1943) and his associates. One important feature of this language is that it distinguishes between multiple senses of natural-language verbs that express being, such as the English word "is." From the vantage point of this language, words like "is" are ambiguous, because sentences containing them can be used to express existence ("There is a Santa Claus"), identity ("Superman is Clark Kent"), predication ("Venus is a planet"), or subsumption ("The wolf is a vertebrate"). In the logical language, each of these senses is expressed in a different way. Yet it is far from clear that the English word "is" really is ambiguous. It could be that it has a single sense that is differently interpreted, or used to

convey different information, depending on the context in which the containing sentence is produced. Indeed, before Frege and Russell, no logician had ever claimed that natural-language verbs of being are ambiguous.
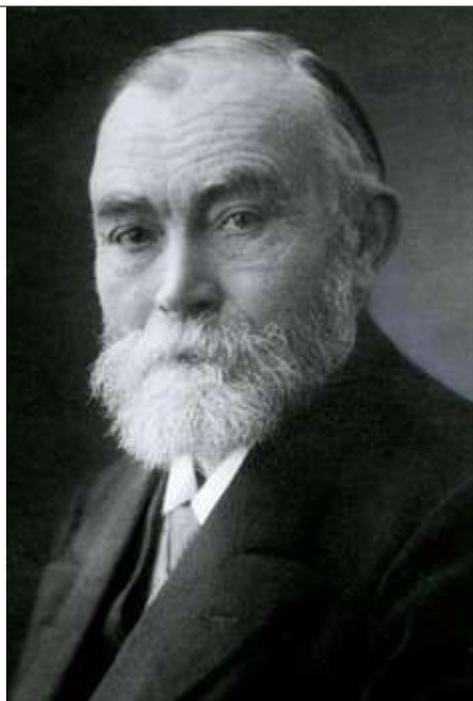
Gottlob Frege.
Courtesy of the Universitatsbibliothek, Jena, Ger.

Another feature of contemporary logical languages is that in them some class of entities, sometimes called the "universe of discourse," is assumed to exist. The members of this class are usually called "individuals." The basic quantifiers of the logical language are said to "range over" the individuals in the universe of discourse, in the sense that the quantifiers are understood to refer to all ($\forall x$) or to at least one ($\exists x$) such individual. Quantifiers that range over individuals are said to be "first-order" quantifiers. But quantifiers may also range over other entities, such as sets, predicates, relations, and functions. Such quantifiers are called "second-order." Quantifiers that range over sets of second-order entities are said to be "third-order," and so on. It is possible to construct interpreted logical languages in which there are no basic individuals (known as "ur-individuals") and thus no first-order quantifiers. For example, there are languages in which all the entities referred to are functions.

Depending upon whether one emphasizes inference and logical form on the one hand or logic translation on the other, one can conceive of the overarching aim of logic as either the study of different logical forms for the purpose of systematizing the study of inference patterns (logic as a calculus) or as the creation of a universal interpreted language for the representation of all logical forms (logic as language).

Logical Systems

Logic is often studied by constructing what are commonly called logical systems. A logical system is essentially a way of mechanically listing all the logical truths of some part of logic by means of the application of recursive rules—i.e., rules that can be repeatedly applied to their own output. This is done by identifying by purely formal

criteria certain axioms and certain purely formal rules of inference from which theorems can be derived from axioms together with earlier theorems. All of the axioms must be logical truths, and the rules of inference must preserve logical truth. If these requirements are satisfied, it follows that all the theorems in the system are logically true. If all the truths of the relevant part of logic can be captured in this way, the system is said to be "complete" in one sense of this ambiguous term.

The systematic study of formal derivations of logical truths from the axioms of a formal system is known as proof theory. It is one of the main areas of systematic logical theory.

Not all parts of logic are completely axiomatizable. Second-order logic, for example, is not axiomatizable on its most natural interpretation. Likewise, independence-friendly first-order logic is not completely axiomatizable. Hence the study of logic cannot be restricted to the axiomatization of different logical systems. One must also consider their semantics, or the relations between sentences in the logical system and the structures (usually referred to as "models") in which the sentences are true.

Logical systems that are incomplete in the sense of not being axiomatizable can nevertheless be formulated and studied in ways other than by mechanically listing all their logical truths. The notions of logical truth and validity can be defined model-theoretically (i.e., semantically) and studied systematically on the basis of such definitions without referring to any logical system or to any rules of inference. Such studies belong to model theory, which is another main branch of contemporary logic.

Model theory involves a notion of completeness and incompleteness that differs from axiomatizability. A system that is incomplete in the latter sense can nevertheless be complete in the sense that all the relevant logical truths are valid model-theoretical consequences of the system. This kind of completeness, known as descriptive completeness, is also sometimes (confusingly) called axiomatizability, despite the more common use of this term to refer to the mechanical generation of theorems from axioms and rules of inference.

Definitory And Strategic Inference Rules

There is a further reason why the formulation of systems of rules of inference does not exhaust the science of logic. Rule-governed, goal-directed activities are often best understood by means of concepts borrowed from the study of games. The "game" of logic is no exception. For example, one of the most fundamental ideas of game theory is

the distinction between the definitory rules of a game and its strategic rules. Definitory rules define what is and what is not admissible in a game—for example, how chessmen may be moved on a board, what counts as checking and mating, and so on. But knowledge of the definitory rules of a game does not constitute knowledge of how to play the game. For that purpose, one must also have some grasp of the strategic rules, which tell one how to play the game well—for example, which moves are likely to be better or worse than their alternatives.

In logic, rules of inference are definitory of the "game" of inference. They are merely permissive. That is, given a set of premises, the rules of inference indicate which conclusions one is permitted to draw, but they do not indicate which of the permitted conclusions one should (or should not) draw. Hence, any exhaustive study of logic— indeed, any useful study of logic—should include a discussion of strategic principles of inference. Unfortunately, few, if any, textbooks deal with this aspect of logic. The strategic principles of logic do not have to be merely heuristic "rules-of-thumb." In principle, they can be formulated as strictly as are definitory rules. In most nontrivial cases, however, the strategic rules cannot be mechanically (recursively) applied.

Rules Of Ampliative Reasoning

In a broad sense of both "logic" and "inference," any rule-governed move from a number of propositions to a new one in reasoning can be considered a logical inference, if it is calculated to further one's knowledge of a given topic. The rules that license such inferences need not be truth-preserving, but many will be ampliative, in the sense that they lead (or are likely to lead) eventually to new or useful information.

There are many kinds of ampliative reasoning. Inductive logic offers familiar examples. Thus a rule of inductive logic might tell one what inferences may be drawn from observed relative frequencies concerning the next observed individual. In some cases, the truth of the premises will make the conclusion probable, though not necessarily true. In other cases, although there is no guarantee that the conclusion is probable, application of the rule will lead to true conclusions in the long run if it is applied in accordance with a good reasoning strategy. Such a rule, for example, might lead from the presupposition of a question to its answer, or it might allow one to make an "educated guess" based on suitable premises.

The American philosopher Charles Sanders Peirce (1839-1914) introduced the notion of "abduction," which involves elements of questioning and guessing but which Peirce

insisted was a kind of inference. It can be shown that there is in fact a close connection between optimal strategies of ampliative reasoning and optimal strategies of deductive reasoning. For example, the choice of the best question to ask in a given situation is closely related to the choice of the best deductive inference to draw in that situation. This connection throws important light on the nature of logic. At first sight, it might seem odd to include the study of ampliative reasoning in the theory of logic. Such reasoning might seem to be part of the subject of epistemology rather than of logic. In so far as definitory rules are concerned, ampliative reasoning does in fact differ radically from deductive reasoning. But since the study of the strategies of ampliative reasoning overlaps with the study of the strategies of deductive reasoning, there is a good reason to include both in the theory of logic in a wide sense.

Some recently developed logical theories can be thought of as attempts to make the definitory rules of a logical system imitate the strategic rules of ampliative inference. Cases in point include paraconsistent logics, nonmonotonic logics, default reasoning, and reasoning by circumscription, among other examples.Most of these logics have been used in computer science, especially in studies of artificial intelligence. Further research will be needed to determine whether they have much application in general logical theory or epistemology.

The distinction between definitory and strategic rules can be extended from deductive logic to logic in the wide sense. Often it is not clear whether the rules governing certain types of inference in the wide sense should be construed as definitory rules for step-by-step inferences or as strategic rules for longer sequences of inferences. Furthermore, since both strategic rules and definitory rules can in principle be explicitly formulated for both deductive and ampliative inference, it is possible to compare strategic rules of deduction with different types of ampliative inference.

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